PHYSICAL REVIEW E, VOLUME 65, 015203(R)

# Fractal fluctuations in quantum integrable scattering

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(Received 18 May 2001; published 20 December 2001)

We theoretically and numerically demonstrate that completely integrable scattering processes may exhibit fractal transmission fluctuations, due to typical spectral properties of integrable systems. Similar properties also occur with scattering processes in the presence of strong dynamical localization, thus explaining recent numerical observations of fractality in the latter class of systems.

DOI: 10.1103/PhysRevE.65.015203 PACS number(s): 05.45.Df, 05.40.−a, 05.45.Mt

Scattering processes dominated by a statistically large number of metastable states display reaction rates that depend on energy or other control parameters in complicated ways. Reaction curves (i.e., curves obtained by plotting cross sections versus the relevant parameters) demand statistical description. This approach, originated in nuclear physics, has given rise to the theory of quantum chaotic scattering [1]. Classical chaotic dynamics plays an important role therein [2], with applications in many fields, including the study of mesoscopic conductance fluctuations [3].

In the presence of complicated reaction curves, a natural question is about their fractality. The latter is not meant in the strict mathematical sense, but rather as a property to be possibly observed on a wide range of resolution scales. Reaction curves probe the location of resonance poles in the complex energy plane, so their fluctuation properties—including fractality—are encoded in the distribution of such poles. In particular, the lack of smoothness required by fractality can only be produced by resonance poles clustering in the vicinity of the real energy axis. This excludes fractal reaction curves in quasiclassical cases when the underlying classical dynamics is completely chaotic (i.e., uniformly hyperbolic), because the quantum resonances are then concentrated away from the real axis. This reflects the exponential decay in time exhibited by such systems over long time scales. A dynamical signature of poles clustering near the real axis is instead slow algebraic decay of the survival probability inside the interaction region. On the classical level, such slow decay is, in particular, exhibited by systems with a mixed phase space, endowed with a hierarchical structure of stable islands [4]. On such grounds, fractal fluctuations were predicted by Ketzmerick for the quasiclassical transmission of electrons through mesoscopic cavities [5]. This prediction has received numerical [6] and experimental [7] support. However, fractal fluctuations have been numerically observed also in two-dimensional tight-binding models of quantum dots, where the relation to classical dynamics is unclear [8], and even in models where this relation is irrelevant [9], due to strong quantum localization on one hand, and to the absence of significant classical critical structures on the other. It therefore appears that fractality of reaction curves is not strictly associated with critical structures in the classical phase space.

The issue of fractality in the remaining major class of dynamical systems, namely, the integrable ones, has not yet been investigated [10]. This is the main purpose of this paper. We first identify special statistical properties of resonance poles, which afford fractal reaction curves. This we do in absolute generality, without assuming integrability or other special properties of the quantum system. We then argue that such properties are optimally exhibited by suitable, completely integrable processes, thanks to a well-known generic property of integrable systems, that their energy levels share some of the properties of a random sequence [11]. We use a textbook scattering problem to confirm our theory with numerical data. Finally, we note that the same properties are also typical of fully chaotic, yet strongly localized systems, and thus we explain the results of Ref. [9].

We consider a weakly open quantum system, with scattering resonances at $E_j - i\Gamma_j/2$ and the real energies $E_j$ arranged in increasing order. The energy dependence of a typical cross section $T(E)$ consists of a smooth background plus a resonant part, which we write in the form

$$T_r(E) = \sum_j c_j \frac{\Gamma_j^2}{(E_j - E)^2 + \Gamma_j^2}, \tag{1}$$

with $c_j$ slowly varying with $j$. We restrict within an energy interval $(E_0 - W, E_0 + W)$, and we assume $\epsilon \ll W \ll E_0$, where $\epsilon$ is the mean level spacing.To perform fractal analysis of the graph of $T_r(E)$ vs $E$, we divide the interval $(E_0 - W, E_0 + W)$ into subintervals $\Delta_k$, $(k = 1,2,\ldots,M)$ of equal size $\delta \propto 1/M$. Upon each subinterval, we pile up squares of side $\delta$, and denote by $N(\delta)$ the total number of squares met by the graph of $T_r(E)$ . Algebraic scaling $N(\delta) \propto \delta^{-f}$ with $f > 1$ between scales $\delta_{min} \ll \delta_{max}$ signals that in between such scales, the graph exhibits a fractal (box-counting) dimension $f$. In order to determine $f$, we compute

$$\frac{\log_{10}[N(\delta)]}{\log_{10}(\delta^{-1})} \approx 2 + \frac{\log_{10}\left[M^{-1}\sum_k \sigma_k(T_r)\right]}{\log_{10}(M)} \approx 2 - \frac{\log_{10}\langle \sigma(T_r)\rangle}{\log_{10}(\delta)},$$

 

where $\sigma_k(T_r) \geqslant 0$ is the maximal excursion of $T_r(E)$ within the $k$th interval. We next assume the following. First, in the given energy interval, the frequency $P(\Gamma)$ of widths less than $\Gamma$ scales like $P(\Gamma) \sim a\Gamma^{1-\alpha}$ at $\epsilon < \Gamma < \overline{\Gamma}$, with $\overline{\Gamma}$ the mean resonance width, and $0 < \alpha < 1$. Second, both $\Gamma_j$ and $E_j$ form uncorrelated sequences. Finally, resonances are strongly overlapped, in the sense that $\epsilon \ll \overline{\Gamma} \ll W$. Then, at $\epsilon \ll \delta \ll \overline{\Gamma}$, the fluctuation of $T_r$ in an interval $\Delta_k$ is mostly due to many tiny resonant peaks that are centered inside the interval and are narrower than $\delta$. Every such peak contributes a Lorentzian term in Eq. (1), and the mean square oscillation of this term as $E$ ranges in $\Delta_k$ is $\sim \delta^{-1} P(\delta)^{-1} \int_0^\delta d P(\Gamma) \Gamma \sim 1$. There are $n(\delta) \sim \epsilon^{-1} \delta P(\delta)$ such peaks; as they contribute uncorrelated oscillations, we estimate $\langle \sigma(T_r) \rangle \sim \sqrt{n(\delta)} \sim \epsilon^{-1/2} \delta^{1-\alpha/2}$. Therefore, $f = 1 + \alpha/2$ in a range $\delta_{min} < \delta < \delta_{max}$, with $\delta_{max} \sim \overline{\Gamma}$, and $\delta_{min}$ roughly estimated by $n(\delta_{min}) \sim 1$, that is, $\delta_{min} \sim (\epsilon/a)^{1/(2-\alpha)}$.

Fluctuating cross sections may also be generated at fixed energy, by varying other parameters, as in the case of magnetoresistance fluctuations in mesoscopic physics. The above analysis carries over to such fluctuations on replacing energy by the relevant parameter, provided that the above assumptions remain satisfied.

The above-described conditions are met in some physically relevant situations. Uncorrelated sequences of energy levels are a distinctive feature of generic integrable systems [11]. We then surmise fractal reaction curves for completely integrable scattering processes, provided they display a slow decay, leading to an inverse power-law distribution of $\Gamma$'s. We shall presently describe an explicit example of such a process.

We consider the quantum dynamics of a particle of a unit mass moving inside the infinite strip $0 \leqslant x \leqslant L_x$ in the $(x,y)$ plane, with tunneling barriers at $y = \pm L_y/2$. The Hamiltonian is

$$H = -\frac{\hbar^2}{2}\Delta + \hbar\sigma\delta(y - L_y/2) + \hbar\sigma\delta(y + L_y/2).$$

We use periodic boundary conditions at $x = 0$, $x = L_x$. The Dirac delta functions enforce additional boundary conditions: $\partial_y \psi(x, y+) - \partial_y \psi(x, y-) = 2\sigma\hbar^{-1}\psi(x,y)$ at $y = \pm L_y/2$. The physical model is a rectangular billiard, whence the particle may escape into semi-infinite leads, by tunneling through the two horizontal sides. For $|y| > L_y/2$ eigenfunctions are $u^{\pm}_{E,m}(x,y) = \phi_m(x)\,\theta^{\pm}_k(y)$, $\phi_m(x) = \exp(2\pi i m x/L_x)$, $\theta^{\pm}_k(y) = A^{\pm}(k)\exp(ik|y|) + B^{\pm}(k)\exp(-ik|y|)$, and $2E/\hbar^2 = k^2 + 4\pi^2 m^2/L_x^2$, with $\pm$ denoting the upper ($y > L_{y/2}$) and the lower ($y < -L_{y/2}$) lead, respectively. For given $E > 0$, there are a finite number of open scattering channels labeled by the integer $m$, $|m| \leqslant \mathrm{Int}(L_x(\pi\hbar)^{-1}\sqrt{E/2})$, and by the lead label $\pm$. The coefficients $A^{\pm}(k)$ are related to $B^{\pm}(k)$ by the scattering matrix. Scattering resonances are located at complex values of energy $z_{n,m} = E_{n,m} - i\Gamma_n/2 = 2\pi^2 m^2 \hbar^2/L_x^2 + k_n^2\hbar^2/2$, $(n = 1,2,\ldots,)$, where $k_n$ are the complex roots of the equations

$$e^{ikL_y} \pm 1 \mp ik\hbar\sigma^{-1} = 0. \tag{2}$$

The following asymptotic formulas can be computed by using Lagrange's theorem on the inversion of analytic functions:

$$E_{n,m} = E^0_{n,m} - \frac{\hbar v_n^2}{L_y \sigma}\left\{\frac{\sigma}{v_n}\arctan\left(\frac{v_n}{\sigma}\right) + O\left(\frac{\hbar}{\sigma L_y}\right)\right\},$$

$$\Gamma_n = \frac{\hbar v_n^3}{L_y \sigma^2}\left\{\frac{\sigma^2}{v_n^2}\log_{10}\left(1 + \frac{v_n^2}{\sigma^2}\right) + O\left(\frac{\hbar}{\sigma L_y}\right)\right\}, \tag{3}$$

where $v_n = n\pi\hbar/L_y$ is the velocity in the $n$th vertical mode of the closed rectangle, and $E^{(0)}_{n,m} = 2\hbar^2\pi^2 m^2/L_x^2 + v_n^2/2$ are the eigenvalues of the closed ($\sigma = \infty$) billiard. The first term on the right-hand side (rhs) of the second Eq. (3) is the decay rate of a classical billiard ball inside the closed rectangle, with velocity $v_n$ in the $y$ direction, and absorption probability at $y = \pm L_y/2$ equal to the transmission coefficient for a plane wave through on a $\delta$ barrier, given by $v_n^2(\sigma^2 + v_n^2)^{-1}$. We assume $\hbar \ll \sigma L_y$ and thereby neglect $O(\hbar/\sigma L_y)$ corrections in Eq. (3). The statistics of real parts of resonances is then similar to the energy-level statistics of an integrable system, with a mean level spacing only different by corrections of order $(\hbar/\sigma L_y)^3$ from that of the closed billiard: $\epsilon = 2\pi\hbar^2(L_x L_y)^{-1}$. The correction to the closed billiard levels in Eq. (3) lifts possible degeneracies due to commensurate geometry.

In an energy interval $(E_0 - W, E_0 + W)$, $\epsilon \ll W \ll E_0$, the smooth (Thomas-Fermi) part of the integrated $P(\Gamma)$ distribution is computed from a microcanonical distribution of classical billiard trajectories at energy $E_0$, each with a decay rate $\Gamma$ (3). This quasiclassical distribution has a mean $\overline{\Gamma}_{E_0} \simeq \hbar\sigma^{-2}L_y^{-1}E_0^{3/2}$, and behaves like $a\Gamma^{1/3}$ at small $\Gamma$, with $a = (\pi^2 E_0/2)^{-1/2}(L_y\sigma^2/\hbar)^{1/3}$. The same behavior may be assumed for the quantal distribution, provided that $E_0 \leqslant \sigma^2$, and that $\epsilon \ll \overline{\Gamma}_{E_0}$. The former condition ensures $\Gamma^{1/3}$ behavior of the quasiclassical distribution in a range of $\Gamma$ values comparable to $\overline{\Gamma}_E$. Together with the latter condition, it ensures that the quasiclassical behavior is observed over a statistically significant number of resonances. Reflecting generic properties of energy spectra of integrable systems, the real parts of resonances, arranged in increasing order, form an essentially uncorrelated sequence. At fixed quantum number $m$, they form an ordered ladder, but the superposition of a large number of different, uncorrelated ladders results in a Poisson-like statistic. On the same grounds we assume uncorrelated resonance widths $\Gamma$, too.

The transmission amplitude at energy $E$ from the lower $m$ channel to the upper $l$ channel is

$$S_{m-,l+}(E) = \frac{k^2(E,m)\hbar^2}{\sigma^2 e^{2ik(E,m)L_y} + [k(E,m)\hbar + i\sigma]^2}\,\delta_{ml}, \tag{4}$$

where $k(E,m) = \sqrt{2E\hbar^{-2} - 4\pi^2 m^2 L_x^{-2}}$. A computation shows that the residue of Eq. (4) at a resonance pole is $\sim -i\Gamma/2$ at small $\Gamma$. Hence, the resonant part of the total transmission coefficient

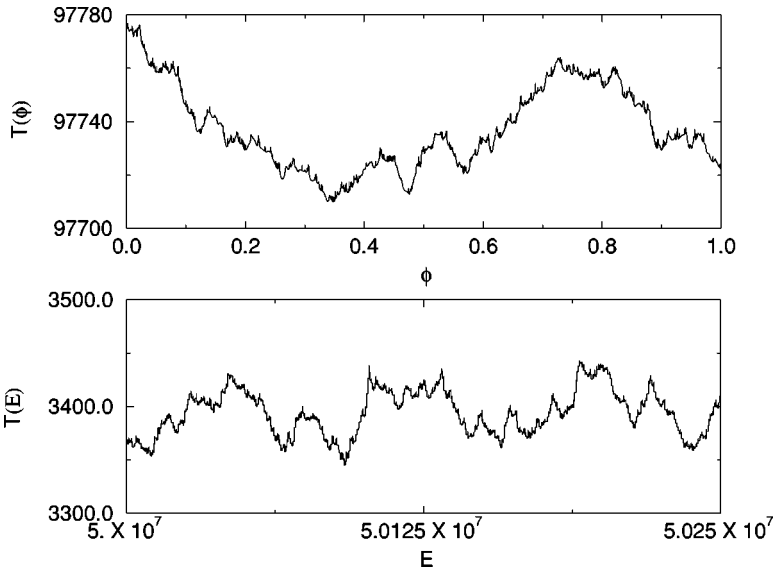


FIG. 1. Total transmission vs flux $\phi$ at fixed energy $E=5\times10^{10}$ (upper plot), and vs energy $E$ at fixed flux $\phi=0$ (lower plot), for $\sigma=3.5\times10^5$ and $\sigma=10^4$, respectively. In both cases $\hbar=1$, $L_x=2$, and $L_y=0.4$. The energy range in the lower plot is $(E_0,E_0+W)$, with $E_0=5\times10^7$ and $W=2.5\times10^5$. Units are arbitrary.

$$T(E)=\sum_m |S_{m-,m+}(E)|^2, \tag{5}$$

(the sum being over all open channels at energy $E$) has the form (1) apart from a slowly varying factor. The smooth part is $\propto E^{5/2}/\sigma^4$. Collecting various estimates, we see that at $\hbar/(\sigma L_y)\ll 1$, $\hbar^2/L_y^2\ll W\ll E_0<\sigma^2$, $E_0^{3/2}L_x\hbar^{-1}\sigma^{-2}\gg 1$, the assumptions of our general argument are satisfied, hence, the graph of $T(E)$ vs $E$ in $(E_0-W<E<E_0+W)$ should be fractal with dimension $f=4/3$, over scales intermediate between $\delta_{min}\sim 1.8\epsilon^{3/4}\overline{\Gamma}_{E_0}^{1/4}$ and $\delta_{max}\sim\overline{\Gamma}_{E_0}$.

A numerically computed graph of $T(E)$ is shown in Fig. 1 (lower). The corresponding fractal analysis is shown in Fig. 2 and fully confirms the theory. In that case, the above estimates for $\delta_{min}$, $\delta_{max}$ give $10^{1.9}$, and $10^{3.9}$ respectively.

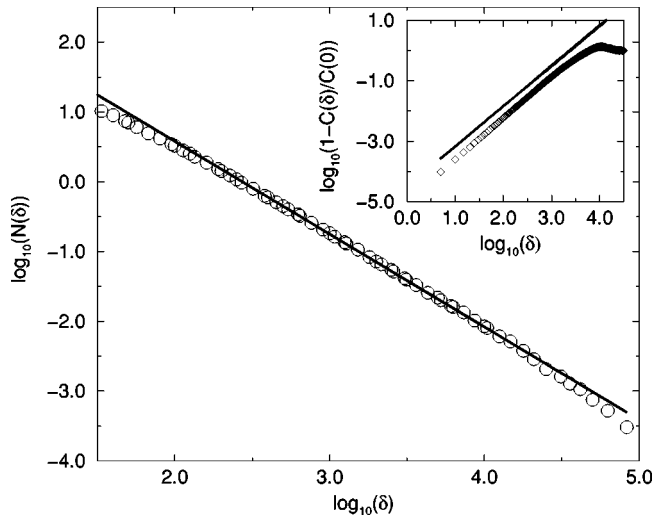


FIG. 2. Fractal analysis of the graph of $T(E)$ vs $E$ shown in the lower Fig. 1. The meaning of $\delta,N(\delta)$ is explained in the text. The straight line corresponds to fractal dimension $f=4/3$. The inset shows the autocorrelation $1-C(\delta)/C(0)$ vs $\delta$. The straight line has slope 4/3.

A generalization of the above model allows for the investigation of parametric fluctuations. It is obtained by using boundary conditions $\psi(0,y)=e^{i\phi}\psi(L_x,y)$. This is equivalent to a particle moving on a cylinder, with the axis in the $y$ direction, enclosing a magnetic flux $\phi$. This problem is still completely integrable. Replacing $m$ by $m-\phi/(2\pi)$ throughout the equations derived at $\phi=0$ yields the corresponding theory. At fixed energy $E$, the total transmission fluctuates as $\phi$ is varied, in the manner illustrated in Fig. 1 (upper). The theory of such fluctuations is completely parallel to the one we have described for fluctuations vs energy at fixed $\phi=0$. Resonances depend on $\phi$, $z_{n,m}=z_{n,m}(\phi)$, and the complex values of $\phi$ solving the equations $z_{n,m}(\phi)=E$ define resonance poles in the complex $\phi$ plane. Omitting computational details, the real parts of such poles are distributed in $[0,2\pi]$ with a mean spacing $\epsilon_\phi\sim\pi^2\hbar\sqrt{2/E}/L_x$. The distribution of their widths $\Gamma_\phi$ behaves similar to $\Gamma_\phi^{1/3}$ at small $\Gamma_\phi$, with a mean $\sim L_x^2\overline{\Gamma}_E\epsilon_\phi/(8\pi^2\hbar^2)$. Our general discussion is thus valid for parametric fluctuations, too. In fact, numerical data shown in Fig. 3 demonstrate the predicted fractal dimension 4/3.

Both for energy dependent and for parametric fluctuations we have computed autocorrelations of the fluctuation graphs. For the case of fluctuations vs energy, such correlations are defined by

$$C(\delta)=\langle T^{(fl)}(E)T^{(fl)}(E+\delta)\rangle_E,$$

where $T^{(fl)}$ is obtained from $T$ by subtracting a smooth, slowly varying part, and the average is taken over the scanned interval of energies. Treating $T^{(fl)}(E)$ as a stationary stochastic process, one easily finds that such correlations behave at small $\delta$ like $C(0)-\text{const.}\times\delta^{2\gamma}$, where $\gamma$ is the scaling exponent of the rms increment of $T(E)$ over intervals of length $\delta$: $\langle|T(E+\delta)-T(E)|^2\rangle_E\propto\delta^{2\gamma}$ (note that subtracting the smooth part does not alter fractional scaling). Generally speaking, the rms *increment* is a quite different quantity from the average *excursion* that enters the definition of the fractal dimension. With the present strong statistical proper-

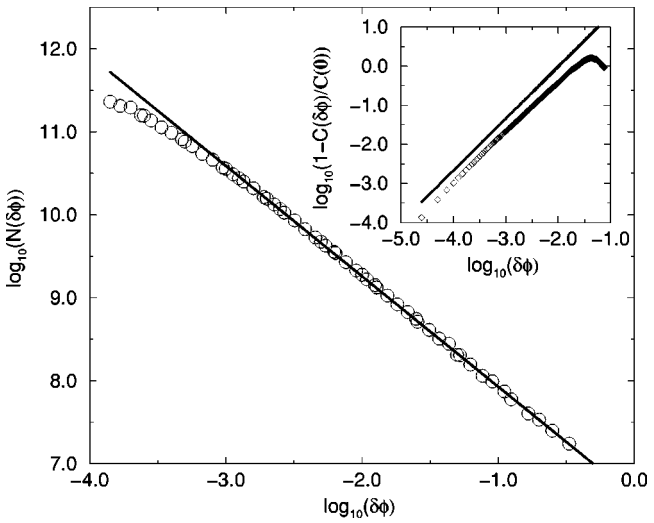


FIG. 3. Fractal analysis of the flux-dependent fluctuations shown in the upper Fig. 1. The straight line corresponds to $f=4/3$. The inset shows the correlation scaling exponent $2\gamma=4/3$.

ties, the two quantities scale in the same way at small $\delta$: $2\gamma=2-\alpha$, so $C(\delta)\sim C(0)-\text{const.}\times\delta^{4/3}$, as confirmed by the numerical data. However, fractional scaling of correlations is not, in general, a sufficient condition for fractality. For instance, correlations in Figs. 2 and in 3 exhibit a 4/3 scaling down to small $\delta$ scales below the fractal range. This is because they are still determined by the statistics of narrow, nonoverlapped individual peaks, which do not produce fractality any more. On the other hand, on increasing $\delta$ correlation functions depart from the predicted fractional behavior already at values well within the fractal range, because a larger statistic is needed for *increments* recorded over a finite $\delta$ grid to sample the distribution of *excursions* over the same grid.

The work reported in this paper hinges on the fully general fact, that fractality of scattering fluctuations is, first and foremost, a matter of complex level statistics. As in the case of real level statistics (closed systems), integrable systems have special properties in this respect, and we have in fact demonstrated fractal reaction curves for a completely integrable process. A few remarks are in order about the relevance of this finding:

(1) In completely integrable scattering processes, there is no mixing of flux between different channels. Similar to pseudorandomness of energy levels, fractality comes of superimposing nonfractal fluctuation patterns from different, uncorrelated channels. Therefore, the nature of the reaction curves depends on the relative weight assigned to different channels.

(2) The present result may also be relevant to quasiintegrable systems, which possess large stable components in their phase space. Such components may contribute a significant set of resonances, produced by tunneling through invariant manifolds, with the statistical properties considered in this paper. Their interplay with critical structures at the border of the stable regions demands careful analysis.

(3) Uncorrelated energy spectra also occur with fully chaotic systems in the regime of strong quantum localization. A prototype system in this class is the kicked rotor, the quasienergy spectrum of which has a Poisson-like statistic in case of strong localization [12]. For this class of systems, the $\Gamma$ (differential) distribution behaves similar to $1/\Gamma$ down to very small scales [13,14]. This explains recent findings [9] of *parametric* fractal fluctuations with dimension 3/2, detected in the *survival probability* at fixed time, on varying a magnetic flux $\phi$.

Support from MURST Research Project ''Chaos and Localization in Classical and Quantum Mechanics'' is gratefully acknowledged.

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